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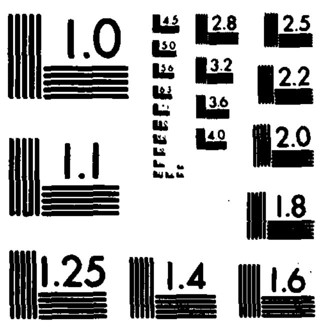
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AFOSR-TR- 80-0991

LEVEL II

FINAL SCIENTIFIC REPORT.

June 1, 1975 to May 31, 1980

AD A091039

Air Force Research Grant AFOSR-75-2839

Input Signal Synthesis for Linear and

Nonlinear System Identification



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# ABSTRACT

The research that has been supported by Air Force Research Grant -AFOSR 75-2839 is described in the Final Scientific Report. This grant was initiated on June 1, 1975 and ended on May 31, 1980. This report provides a comprehensive review of accomplishments and a chronological bibliography of all publications resulting from the support provided by the grant.

## 1. Objectives and Summary of the Research

In the proposal that led to this research grant, the objective of the research was summarized in the following manner.

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"The proposed study will consider the general problem of optimal input signal synthesis for dynamic systems. Different optimality criteria are to be investigated and compared. The primary emphasis in the study shall be on the development of feedback control policies for specific criteria. The improvements possible through the use of suitably chosen input signals will be investigated."

With these objectives, a large variety of investigations have been pursued. Detailed presentations of much of this work is described in the papers listed in Section 2 of this Final Report and shall not be repeated here. Let us summarize the major characteristics of these studies.

1. Optimality criteria. A substantial portion of the effort during the grant period was directed to the investigation of interrelationships between different performance indices. Particular emphasis was placed on the Bhattacharyya distance and on different mappings of the Fisher information matrix. The results of these studies are described in the doctoral dissertation by Upadhyaya and in papers (1, 5, 8, 9) as listed in Section 3.

2. Feedback policies. The study of different performance criteria included the comparison of open-loop and closed-loop control policies. These studies initially produced surprising and disappointing results. In particular, little performance improvement was observed for feedback policies. It is

only recently that a better understanding of these results has been obtained. These results are discussed in Section 2 of this report.

3. Interaction of input signals and identification algorithms. Numerical studies indicated considerable interaction between the performance of "optimal" input signals and the specific identification algorithm. These interactions led to inconsistent results that made general conclusions difficult to obtain. These inconsistencies led to the consideration of the problem from a general perspective which has yielded important insights into the input signal synthesis/system identification/stochastic control problem. These results, which have become available during the last year of the grant, are described in Section 2. This work appears to more than fulfill the objectives of the results and will continue during the coming year.

4. Dual control considerations. It was observed in the original proposal that the design of input signals can be regarded as a restricted version of the stochastic control problem. That is, the control of a stochastic system has two general features. Certainly, the control must achieve some prescribed control objective for the system. But, also, it generally must induce some "learning" about system parameters and variables. These features have come to be known as "dual aspects" of stochastic control theory. Our studies have concentrated on the learning aspect but some investigations of the general stochastic control problem were undertaken. These studies are described in papers (2, 11, 12, 15, 18).

5. Numerical optimization methodology. The formulation of optimal input signal synthesis problems leads to complex dynamic optimization problems which must be solved numerically. This requirement led to some work involving optimization procedures suitable for solving large-scale problems that have

arisen in the course of this research. This work is described in the doctoral dissertation by Koble and in papers (3, 4, 6, 7, 16).

## 2. A Unifying Approach to the Analysis of the Input Signal Synthesis Problem

Published results supported by this grant have been summarized in the preceding paragraphs. This work leaves many unanswered questions. During the final year of the grant period, attention has been directed to examining the problem from a general perspective with the goal of obtaining insights into these difficult questions. We feel that this objective has been achieved although considerable work remains. This effort will be completed during the coming year with the completion of the doctoral research of J. R. Miller and the preparation of several papers summarizing the results. In this section we shall present the basic approach and discuss important insights.

For this discussion, we shall restrict our attention to a scalar system. The results generalize to multivariable systems but notational and technical details tend to cloud the basic issues.

Consider the following stochastic system with scalar state $x(t)$ and scalar measurement $z(t)$ described by

$$x(t+1) = a\,x(t) + u(t) + w(t) \tag{1}$$

$$z(t) = x(t) + v(t)\,, \qquad t = 1, 2, \cdots \tag{2}$$

The input signal $u(t)$ can be regarded as a function of time or as a feedback signal generated from the measurements $\{z(1), z(2), \ldots, z(t)\}$. The initial state $x(1)$ is Gaussian with mean $\hat{x}(1)$, and variance $P(1)$. The sequences

$\{w(t)\}$, $\{v(t)\}$ are stationary Gaussian white noise with zero mean and variances Q, R, respectively. The noise sequences are assumed to be mutually independent and independent of the initial state x(1).

Suppose that the transition variable a is a Gaussian random variable with mean $\hat{a}_1$ and variance $\Sigma(1)$. Further, a and x(1) are assumed to be independent.

The system (1)-(2) can be regarded as a prototypical model of a system that is identifiable, controllable, and observable. Our discussion generalizes naturally to multivariable systems having these properties.

The combined system identification/state estimation problem for the system (1)-(2) is easily stated. From the measurements $\{z(1),\ z(2),\ \ldots,\ z(t)\} \stackrel{\Delta}{=} \underline{z}^t$ and knowledge of the inputs $\{u(1),\ u(2),\ \ldots,\ u(t-1)\} \stackrel{\Delta}{=} \underline{u}^{t-1}$, we want to estimate the state x(t), and to identify the transition parameter a. Basically, this is a nonlinear estimation problem since a and x(t) appear multiplicatively in (1). As a result, there is no closed-form for the estimator of [a, x(t)]. Instead, estimators have been proposed which are motivated by reasonable arguments and ad hoc approximations. For example, the extended Kalman filter has been used for this problem with varying success. Ljung recently proposed a modification of the EKF for which he claims to prove convergence. The output of any estimator/identifier is regarded as numbers represented as

$$\hat{a}_t = h(\underline{z}^t,\ \underline{u}^{t-1},\ t)$$

$$\hat{x}_t = g(\underline{z}^t,\ \underline{u}^{t-1},\ t)\ .$$

The input signal synthesis problem arises by determining the input sequence $\underline{u}^{t-1}$ in a manner designed to enhance the quality of the estimates $\hat{a}_t$, $\hat{x}_t$. For example, the input sequence might be defined as "best" if it causes the estimates $\hat{a}_t$, $\hat{x}_t$ to achieve the smallest error variance.

Generally, it is stated that it is not possible to determine the mean-square estimation error. Consequently, other performance measures have been introduced as reasonable substitutes. The Cramer-Rao lower bound as given by the inverse of the Fisher information matrix, can be computed. As a result some scalar mapping of this matrix is often used as a measure of optimality. From the experience in this effort, these approaches have produced many results having questionable utility. Furthermore, the quality of the results that are obtained for these and other performance measures has been found to depend in a nontrivial fashion upon the specific form of the estimation/identification algorithm.

The choice of the optimality criteria and the choice of the estimation/identification algorithm are necessarily made in a somewhat arbitrary manner. This arbitrariness has haunted the investigation and led to considerable frustration in many respects. As a result, the effort during the past year has been directed toward an analysis of the general problem from a viewpoint that permits avoidance of these arbitrary choices. This analysis is made possible by adopting the Bayesian approach described in the remainder of this section.

The random variables in (1)-(2) have been assumed to be Gaussian.* However, the plant nonlinearity $a\,x(t)$ causes $x(t+1)$ to be nonGaussian. Using Bayes' rule, we can obtain the a posteriori density for a, x(t) given the input $\underline{u}^{t-1}$ and output $\underline{z}^{t}$. In particular, it can be shown that the general form of Bayes' rule has the following general recursive description.

Filtering Density

$$f\left(a, x(t) \mid \underline{z}^{t}, \underline{u}^{t-1}\right) = \frac{f_V\left(z(t) \mid x(t)\right) f\left(a, x(t) \mid \underline{z}^{t-1}, \underline{u}^{t-1}\right)}{f\left(z(t) \mid \underline{z}^{t-1}, \underline{u}^{t-1}\right)} \qquad (3)$$

where

$$f\left(z(t) \mid \underline{z}^{t-1}, \underline{u}^{t-1}\right) = \int f_V\left(z(t) \mid x(t)\right) f\left(a, x(t) \mid \underline{z}^{t-1}, \underline{u}^{t-1}\right) dx(t)$$

At $t = 1$, we must define

$$f\left(a, x(1) \mid \underline{z}^{0}, \underline{u}^{0}\right) \triangleq f\left(a, x(1)\right) .$$

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*We use the following convention for a n-dimensional Gaussian random vector $\underline{x}$ with mean $\underline{\mu}_x$ and covariance and matrix $\Sigma_x$.

$$N_X(\underline{\mu}_x, \Sigma_x) = (2\pi)^{-n/2} (\det \Sigma_x)^{-1/2} \exp\left\{-\frac{1}{2} (\underline{x} - \underline{\mu}_x)^T \Sigma_x^{-1} (\underline{x} - \underline{\mu}_x)\right\} .$$

Prediction Density

$$f\left(a, x(t+1) \mid \underline{z}^t, \underline{u}^t\right) = \int f_W\left(x(t+1) - ax(t)\right.$$

$$\left. - u(t)\right) f\left(a, x(t) \mid \underline{z}^t, \underline{u}^{t-1}\right) dx(t) \ . \tag{4}$$

These densities assume a more specific form when the Gaussian assumptions regarding the initial state $x(1)$, the transition parameter a, and the noise sequences $\underline{w}^t$, $\underline{v}^t$ are introduced. We shall state the results and discuss the information that is provided by their examination during the remaining paragraphs of this section.

Prediction Density/Gaussian a priori

$$f\left(a, x(t+1) \mid \underline{z}^t, \underline{u}^t\right) = f(a \mid \underline{z}^t, \underline{u}^t) \ N_{t+1|t} \left(\hat{x}_{t+1|t}, P(t+1 \mid t)\right) \tag{5}$$

where

$$\hat{x}_{t+1|t} = a\hat{x}_{t|t} + u(t)$$

$$P(t+1|t) = a^2 P(t \mid t) + Q \ .$$

Filtering Density/Gaussian a priori

$$f\left(a, x(t) \mid \underline{z}^t, \underline{u}^{t-1}\right) = f(a \mid \underline{z}^t, \underline{u}^{t-1}) \ N_{t|t} \left(\hat{x}_{t|t}, P(t \mid t)\right) \tag{6}$$

where

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K(t)\ [z(t) - \hat{x}_{t|t-1}]$$

$$K(t) = P(t\,|\,t-1)\ [P(t\,|\,t-1) + R]^{-1}$$

$$P(t\,|\,t) = P(t\,|\,t-1) - K(t)\ P(t\,|\,t-1)$$

$$f(a\,|\,\underline{z}^{t},\ \underline{u}^{t}) = f(a\,|\,\underline{z}^{t},\ \underline{u}^{t-1})$$

$$= c_t\ N_{I_t}\big(0,\ \pi(t\,|\,t-1)\big)\, f(a\,|\,\underline{z}^{t-1},\ \underline{u}^{t-1}) \tag{7}$$

where

$$i_t = z(t) - \hat{x}_{t|t-1}$$

$$\pi(t\,|\,t-1) = P(t\,|\,t-1) + R$$

$$1/c_t = \int N_{I_t}\big(0,\ \pi(t\,|\,t-1)\big) f(a\,|\,\underline{z}^{t-1},\ \underline{u}^{t-1})\ da\ .$$

<u>Initial</u> <u>Conditions</u>:

$$f(a\,|\,\underline{z}^{0},\ \underline{u}^{0}) = f(a) = N_A\big(\hat{a}_1,\ \Sigma(1)\big)$$

$$f\big(a, x(1)\big) = f(a)\, f\big(x(1)\big)$$

$$f\big(x(1)\big) = N_1\big(\hat{x}(1),\ P(1)\big)$$

Equations (5), (6), and (7) provide a general basis for the study of the state estimation/system identification/input signal synthesis problem. Let us summarize some important observations regarding these equations.

(1) We have made no assumptions regarding the form of the estimation and identification algorithms. As discussed below, the development does suggest a natural and intuitively appealing structure for an estimation algorithm.

(2) No assumptions regarding the source of the input sequence have been made other than to assume u(t) is a known function of time (i.e. an open-loop policy) or a known function of the output sequence $\underline{z}^t$ (i.e. a closed-loop policy).

(3) The Kalman filter provides a natural basis for updating the parameters of the conditional densities $N_{t+1|t}$ and $N_{t|t}$ in (5) and (6). Note that no modification of the state estimator is suggested by the relations (5) and (6) in distinction with Ljung's approach in modifying the EKF.

To expand upon this point, consider the maximum a posteriori estimator of x(t) and a as obtained by considering (5)-(7). These estimators must satisfy the condition

$$\frac{\partial f\left(a,\ x(t)\ |\ z^t,\ u^t\right)}{\partial x(t)} = 0$$

$$\frac{\partial f\left(a,\ x(t)\ |\ z^t,\ u^t\right)}{\partial a} = 0$$

The first condition is seen to be satisfied by choosing

$$\hat{x}_{MAP}(t \mid t) = \hat{x}_{t \mid t}\left(\hat{a}_{MAP}(t \mid t)\right)$$

$$\hat{x}_{MAP}(t \mid t - 1) = \hat{x}_{t \mid t - 1}\left(\hat{a}_{MAP}(t \mid t)\right)$$

Thus, the Kalman filter provides the MAP estimator of the state. Note, however, that it must be evaluated with the MAP estimator of a. This estimator is not readily obtained in a closed-form from (5)-(7). We shall discuss this problem below.

(4) The determination of the MAP estimator of a using (5)-(7) leads to complicated algebraic equations that must be solved since a enters $f\left(a, x(t) \mid z^t, u^t\right)$ through $f(a \mid z^t, u^t)$, through $\hat{x}_{t \mid t}$, and through $P(t \mid t)$. For the purposes of this analysis, it is important to recognize that is more useful to defer the question of obtaining a suitable estimator. Instead, we want to examine the characteristics of the density function itself. Thus, we shall consider the numerical evaluation of (5)-(7) for a range of parameter values. Some representative results are presented below that illustrate the insights that can be gained.

(5) The input signal enters the densities only through the updating-relation for $\hat{x}_{t+1 \mid t}$ in (5). But it alters both $x_{t+1}$ and $\hat{x}_{t+1 \mid t}$ in precisely the same manner. Thus, the input only shifts $N_{t+1 \mid t}$ but does not otherwise alter the density function as it involves the state. But this is a manifestation of the well known Separation Theorem for linear, Gaussian systems and quadratic performance indices.

(6) The input signal can have a profound effect upon the nature of the density function relative to the transition parameter a. However, it enters in a subtle fashion. The input affects the propagation of the state which is then reflected through the measurement and, more significantly, through the innovation sequence $i_t$.

To describe the behavior of the density function and to illustrate the influence of the input signal, it is useful to consider numerical examples. These examples are representative of more extensive studies and demonstrate the utility of the general approach.

For this discussion, assume that the plant and measurement noise variances, Q and R, are equal to 4. The a priori mean and variance for the transition parameter a will be taken to be 0.4 and 0.16, respectively. The initial state has mean zero and variance 100. This completes the definition of the parameters of the problem.

Consider the system identification/state estimation problem when there are no inputs (i.e. $u(t) \equiv 0$ for all t). By computing the a posteriori density $f\left(a, x(t) \mid z^t, u^{t-1}\right)$ we can consider several questions. For this illustration, we shall present results regarding the following questions.

1. How is the a posteriori density affected by the true value of the transition parameter a?

2. How does the extended Kalman filter perform as an estimator of a and x(t)?

3. Does the a posteriori density converge to a Gaussian-like distribution?

4. Do different noise realizations affect the maximum of the distribution or the spread of the distribution?

5. How accurate an estimator of the a and $x(t)$ is provided by the maxima of the a posteriori density?

These questions merely scratch the surface regarding the types of investigation that are made possible through the evaluation of the a posteriori density relation (5)-(7). We know of no other approach that permits us to investigate even these questions in the general and insightful manner that our results provide.

Let us consider results that speak to the preceding questions. We shall provide printer plots that provide graphical descriptions of the results.

Case 1:

$$a_{TRUE} = 0.4$$

$$x_{TRUE}(1) = -5 \ .$$

In Figure 1, a two-dimensional plot of $f\left(a, x(1) \mid \underline{z}^1\right)$ is provided for specific values of a and $x(1)$. The transition parameter a is discretized (horizontal axis) in units of 0.1 whereas the state (vertical axis) is discretized in units of 1. The number in each cell represents the value of the density function at the grid point except for the appropriate normalization constant (approximately 1000 in this instance). For this plot, a single measurement $z(1)$ has been processed. As is seen by inspecting equation (5), this measurement provides information only regarding $x(1)$ and not a. Thus, the marginal density $f\left(x(1) \mid z(1)\right)$ displayed in the right hand column is no longer

the a priori Gaussian whereas the marginal density displayed in the upper row is given as $f\left(a \mid z(1)\right) = f(a)$.

We repeat this type of plot for 500, 1000, and 1500 samples in Figures 2, 3, 4 and can use these plots to discuss some of the earlier questions.

1. Observe that the densities retain their unimodality. In fact, they remain reasonably symmetric and Gaussian-like. Further analysis indicates that a Gaussian distribution is a good approximation for the marginal densities except for small (i.e. less than 50 samples) sample sizes.

2. The true value of the transition parameter a is 0.4. The true value of the state changes at each sample time and is indicated on the plots. Note that the peak of the density function provides a reasonable estimate of a and $x(t)$ but is certainly not error-free. The distribution retains considerable "spread" (i.e. nontrivial variance) even after processing 1500 samples. The error in $\hat{a}_{MAP}$, $\hat{x}_{MAP}(t)$ is consistent with the variance of the density function.

3. The observation that the density function exhibits a substantial variance after 1500 samples implies slow convergence for any identification algorithm. It is useful to consider the behavior of the marginal density $f(a \mid \underline{z}^t)$. This density is plotted in Figure 5. These densities change very slowly and the results suggest that convergence in any practical situation (finite sample sizes) is not to be expected. The standard deviation for these densities is shown on the Figure. When one notices that the magnitude of the state lies generally within the range of the

measurement noise (i.e. $\sqrt{R} = 2$), it should not be surprising that the measurements provide little information about the transition parameter a.

Case 2:

$$a_{TRUE} = 0.9$$
$$x_{TRUE}(1) = -5 .$$

Consider now the effect of a larger value for the transition parameter a. Instead of the overdamped system in Case 1, the system is assumed to be underdamped with a transition parameter value that is near to the stability boundary. In investigating the convergence of many identification algorithms, considerable attention is given to cases in which a is near one. Our analysis indicates that one should expect better performance from a suitable algorithm as the stability boundary is neared. This observation is illustrated in the two dimensional plots of $f\left(a, x(t) \mid \underline{z}^t\right)$ provided in Figure 6 and confirmed in Figure 7 with plots of the marginal density $f(a \mid \underline{z}^t)$. We note that the marginal density appears to be converging to a Gaussian distribution with a diminishing variance. The maximum of the density function provides an accurate estimate of a. Also, the EKF estimate is well-behaved and reasonable.

Case 3:

$$a_{TRUE} = 0.1$$
$$x_{TRUE}(1) = -5 .$$

The results presented above relate to a single realization of the plant and measurement noise sequences. Let us examine the influence that different noise realizations can have on the densities. As noted above, the behavior becomes more erratic as $|a|$ approaches zero. To emphasize the possible influence of the noise realization, we shall assume that the true value of the transition parameter is 0.1. The marginal density after 500 samples is presented in Figures 8 for three different noise realizations. Except for the noise realization, the parameters of the runs are identical.

The noise realization is seen to have a substantial influence on the location of the peak of the density. The variance is not affected to as great an extent and the Gaussian-like character of the density is not changed, at least not through any cursory examination. The behavior of the EKF is very sensitive to the realization.

This completes our illustration of the system identification/state estimation problem in the absence of any input signal. As a minimum, one should realize that system identification in this context is very difficult, particularly for overdamped systems which are driven by unobservable white noise. In fact, convergence of an identifier may be illusory in the sense that the variance of the parameter estimate tend to zero very slowly (if at all).

The identification is made difficult because we have been considering a stable system. The state in the absence of plant noise vanishes exponentially. With plant noise the behavior of the state tends to white noise as the transition parameter a tends to zero. Thus, it becomes increasingly difficult to extract information about a and about x(t) from the noisy information.

The system identification problem becomes more tractable when the system is driven by known inputs. This point is illustrated by considering our example with a variety of input signals having the same energy. The noise realizations are the same in each case as are all parameters of the problem. We saw earlier that $a_{TRUE} = 0.4$ led to a difficult identification problem and we shall reconsider this case.

The effect of known input signal is compared with the zero input case. Four types of input signals are considered.

1. Impulse every ten samples.

$$u(t) = \begin{cases} 4\sqrt{10}, & t = 10i + 1, \quad i = 0, 1, 2, \cdots \\ 0, & \text{all other } t \end{cases}$$

2. *Pseudo random sequence*

$$n(t) = 4 \,\mathrm{sgn}\big(n(t)\big)$$

where $n(t)$ is a sample chosen from a uniformly-distributed white noise generator.

3. White noise

$$u(t) = \nu(t)$$

where $\nu(t)$ is sample from a Gaussian white noise sequence with zero mean and variance 4.

4. Sinusoidal input

$$u(t) = \frac{160}{\pi} \sin(\pi t/5)$$

The energy in the input signal for a 10 sample interval is the same

$$E = \sum_{i=1}^{11} u^2(i) = 160$$

and is equal to four times the average energy in the plant noise and in the measurement noise.

The joint and marginal densities for the zero input and for the four input signals are shown in Figures 9, 10 and 11 after processing 500 measurement samples. Some interesting conclusions are apparent.

1. The input signals force the state to assume values that are generally larger than the noise signals (i.e. the output signal-to-noise ratio is increased). Consequently, the marginal density $f(a \mid \underline{z}^k)$ has a much smaller variance. This is apparent by considering the variances displayed on Figures 10 and 11. The variance is reduced approximately by one-third through the introduction of input signals.

2. The greatest reduction in variance comes from the use of a sinusoidal signal. This conforms with analytical results published in earlier studies conducted under the auspices of this grant. However, the results are not very sensitive to the form of the input signal. The primary influence is the input signal energy.

3. The extended Kalman filter performs satisfactorily. In fact, it serves as a reasonable approximation of a maximum a posteriori estimator. This improvement in behavior reflects our earlier comments regarding the apparent coupling between the identification algorithm and the input signal.

4. The insensitivity of the density behavior to the form of the input signal provides insight into the earlier comments regarding the unsatisfactory behavior of feedback signals. Although no feedback signals are considered in these results, it has become apparent that the identification is enhanced primarily by increasing the signal-to-noise ratio. On the average this is accomplished as well by open-loop signals as by feedback signals.

Let us summarize the preceding discussion. By taking a Bayesian approach, the a posteriori joint density for the system parameters and for the state given the input and output signals can be determined. This density provides a sufficient statistic for the problem and permits a foundation for considering the performance of system identification/state estimation algorithms and for assessing the input of prescribed input signals. This fundamental approach has been lacking in previous analyses. It certainly provides a perspective on the earlier work produced during this grant and much of the published literature. Papers describing these recent results will be forthcoming during the next few months.

## 3. Publications and Reports Supported by the Grant

The research that has been supported partially or wholly by this grant has led to the publication of several reports and papers. We list these documents below. The Principal Investigator has completed a book, listed below, that is to be published in the near future. While not a specific activity of this research program, the Principal Investigator wants to acknowledge his appreciation for the support provided by AFOSR for his research activities during the past several years. This support has contributed substantially to the author's development and, at least indirectly, to the completion of the book. His appreciation has been acknowledged specifically in the Preface.

### Book

H. W. Sorenson, Parameter Estimation: Principles and Problems, Marcel Dekker Publishing Company, New York, 1980.

### Ph.D. Dissertations

1. B. R. Upadhyaya, "Synthesis of Input Signals in Parameter Estimation Problems," University of California, San Diego, December 1975.

2. H. M. Koble, "The Problem Manipulation/Solution Strategy Concept and Its Application to Large-scale Deterministic Optimal Control Problems," University of California, San Diego, September 1980.

Papers

1. H. W. Sorenson, B. R. Upadhyaya, "Synthesis of Stationary, Stochastic Inputs in Identification Problems," Proceedings of Sixth Symposium on Nonlinear Estimation Theory, San Diego, 1975, pp. 249-253.

2. H. W. Sorenson, "An Overview of Filtering and Stochastic Control in Dynamic Systems," Chapter 1 in Control and Dynamic Systems: Advances in Theory and Applications, edited by C. T. Leondes, Academic Press, 1976, pp. 1-63.

3. H. W. Sorenson, "An Introduction to Nonlinear Programming - Part I: Necessary and Sufficient Conditions," Computers and Electrical Engineering, 3, 1976, pp. 1-32.

4. H. W. Sorenson, "An Introduction to Nonlinear Programming - Part II: The Linear Programming Problem," Computers and Electrical Engineering, 3, 1976, pp. 127-157.

5. H. W. Sorenson, B. R. Upadhyaya, "Bayesian Discriminant Approach to Input Signal Selection in Parameter Estimation Problems," Information Sciences, 12, 1977, pp. 60-91.

6. H. W. Sorenson, "An Introduction to Nonlinear Programming - Part III: Search Procedures for Unconstrained Minimization," Computers and Electrical Engineering, 3, 1976, pp. 239-269.

7. H. W. Sorenson, H. M. Koble, "An Introduction to Nonlinear Programming - Part IV: Numerical Methods for Constrained Minimization," Computers and E ectrical Engineering, 3, 1976, pp. 347-386.

8. H. W. Sorenson, B. R. Upadhyaya, "Synthesis of Linear Stochastic Signals in Identification Problems," Proceedings of the 1976 IEEE Conference on Decision and Control, Clearwater Beach, Florida, December 1976, pp. 941-946.

9. B. R. Upadhyaya, H. W. Sorenson, "Synthesis of Linear Stochastic Signals in Identification Problems," Automatica, 13, 1977, pp. 615-622.

10. H. W. Sorenson, "Approximate Solutions of the Nonlinear Filtering Problem," Proceedings of the 1977 Conference on Decision and Control, New Orleans, Louisiana, December 1977.

11. H. W. Sorenson, V. S. Samant, "A Computational Method for Stochastic Optimization," Proceedings of the 1977 Conference on Decision and Control, New Orleans, Louisiana, December 1977.

12. V. S. Samant, H. W. Sorenson, "Stochastic Optimal Control Via Stochastic Programming," Proceedings of the Sixth Annual Pittsburgh Conference on Modelling and Simulation, April 1975.

13. A. V. Sebald, A. Haddad, "State Estimation for Singularly Perturbed Systems with Uncertain Perturbation Parameters," IEEE Transactions on Automatic Control, AC23, No. 3, June 1978.

14. A. V. Sebald, "State Estimation in Systems Driven by Poisson Processes with Unknown Arrival Rates," Proceedings of the 1977 Conference on Decision and Control, New Orleans, Louisiana, December 1977.

15. A. V. Sebald, "Toward a Computationally Efficient Optimal Solution to the LQG Discrete-Time Dual Control Problem," IEEE Trans. Auto. Cont., AC-24, 1979, pp. 535-540.

16. C. S. Berger, "Optimal Input for Stochastic Linear Discrete Systems as a Function of Input and Output Variables," Elect. Letters, 15, 1979, pp. 360-361.

17. C. S. Berger, "Synthesis of Input Signals for Parameter Identification Using Moving-Average Filter," Proc. IEEE, 126, 1979, pp. 1311-1315.

18. H. W. Sorenson, D. L. Alspach, "Gaussian Sum Solution of the Nonlinear Filtering and Stochastic Control Problem," Chapter in Nonlinear Filtering and Estimation Theory (edited by E. B. Stear), Marcel Dekker Publishing Company, New York, 1980.

□ - Denotes maximum of $f(a, x(1)|\underline{z}(1))$. ○ - Denotes EKF estimate

| State x(1) \ f(a) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 8 | 13 | 21 | 33 | 46 | 60 | 74 | 87 | 96 | 99 | 96 | 87 | 74 | 60 | 46 | 33 | 21 | 13 | 8 | 4 | $f(x(1)\|z(1))$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| -20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| -9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 18 |
| -8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 4 | [illegible] | 5 | 5 | 5 | 4 | 4 | 3 | 2 | 2 | 1 | 1 | 0 | 0 | 47 |
| -7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | [illegible] | 5 | 6 | 8 | 9 | 10 | 10 | 10 | 9 | 3 | 6 | 5 | 3 | 2 | 1 | 1 | 0 | 100 |
| -6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 5 | 7 | 10 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | 12 | 10 | [illegible] | 5 | 4 | 2 | 1 | 1 | 160 |
| -5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 3 | 4 | 7 | 9 | 12 | 15 | [illegible] | [illegible] | $\boxed{\text{[illegible]}}$ | [illegible] | [illegible] | 15 | [illegible] | 9 | 7 | 4 | 3 | 2 | 1 | 200 |
| -4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 3 | 4 | 6 | 9 | 12 | 14 | 17 | 18 | 19 | 18 | 17 | 14 | 12 | 9 | 6 | 4 | 3 | 2 | 1 | 191 |
| -3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 3 | 5 | 6 | 8 | 10 | 12 | 13 | 14 | 13 | 12 | 10 | 8 | 6 | 5 | 3 | 2 | 1 | 1 | 136 |
| -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 8 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 1 | 0 | 82 |
| -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 1 | 1 | 0 | 0 | 0 | 33 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 11 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

-1.0 -0.5 0 0.5 1.0

Transition parameter a

$a_{TRUE} = 0.4$

$x_{TRUE}(1) = -3.4$

Figure 1: A posteriori densities, $f(a, x(1)|z(1))$, $f(a)$, $f(x(1)|z(1))$.

Note: Density values are scaled by a factor of 1000.

□ - Denotes maximum of $f\left(a, x(500) \mid \underline{z}^{500}\right)$ ○ - Denotes EKF estimate

| State x(500) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $f\left(x(500) \mid \underline{z}^{500}\right)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f\left(a \mid \underline{z}^{500}\right)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 9 | 211 | 632 | 148 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| -20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 3 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 |
| -4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6 | 17 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 27 |
| -3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 21 | 57 | 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 94 |
| -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 44 | 129 | 30 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 205 |
| -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 57 | $\boxed{174}$ | 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 276 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 48 | 145 | 35 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 230 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 24 | 75 | 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 118 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 | 24 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 37 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 5 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | | | | -1.0 | | | | | -0.5 | | | | | 0 | | | | | 0.5 | | | | | 1.0 | | | | | |

Transition parameter a

$a_{TRUE} = 0.4$

$x_{TRUE}(500) = -0.9$

Figure 2: A posteriori densities $f\left(a, x(500) \mid \underline{z}^{500}\right)$, $f\left(a \mid \underline{z}^{500}\right)$, $f\left(x(500) \mid \underline{z}^{500}\right)$

Note: Density values have been scaled by a factor of 1000.

☐ - Denotes maximum of $f\left(a, x(1000) \mid \underline{z}^{1000}\right)$ ◯ - Denotes EKF estimate

| State x(1000) | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | $f\left(x(1000) \mid \underline{z}^{1000}\right)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(a \mid \underline{z}^{1000})$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 | 646 | 296 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| -20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 |
| -3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 17 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 |
| -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 59 | 28 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 88 |
| -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 131 | 60 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 193 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | $\boxed{178}$ | 81 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 261 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 149 | 68 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 219 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 78 | 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 115 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 25 | 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 37 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 7 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | | | | -1.0 | | | | | -0.5 | | | | | 0 | | | | | 0.5 | | | | | 1.0 | | | | | 950 |

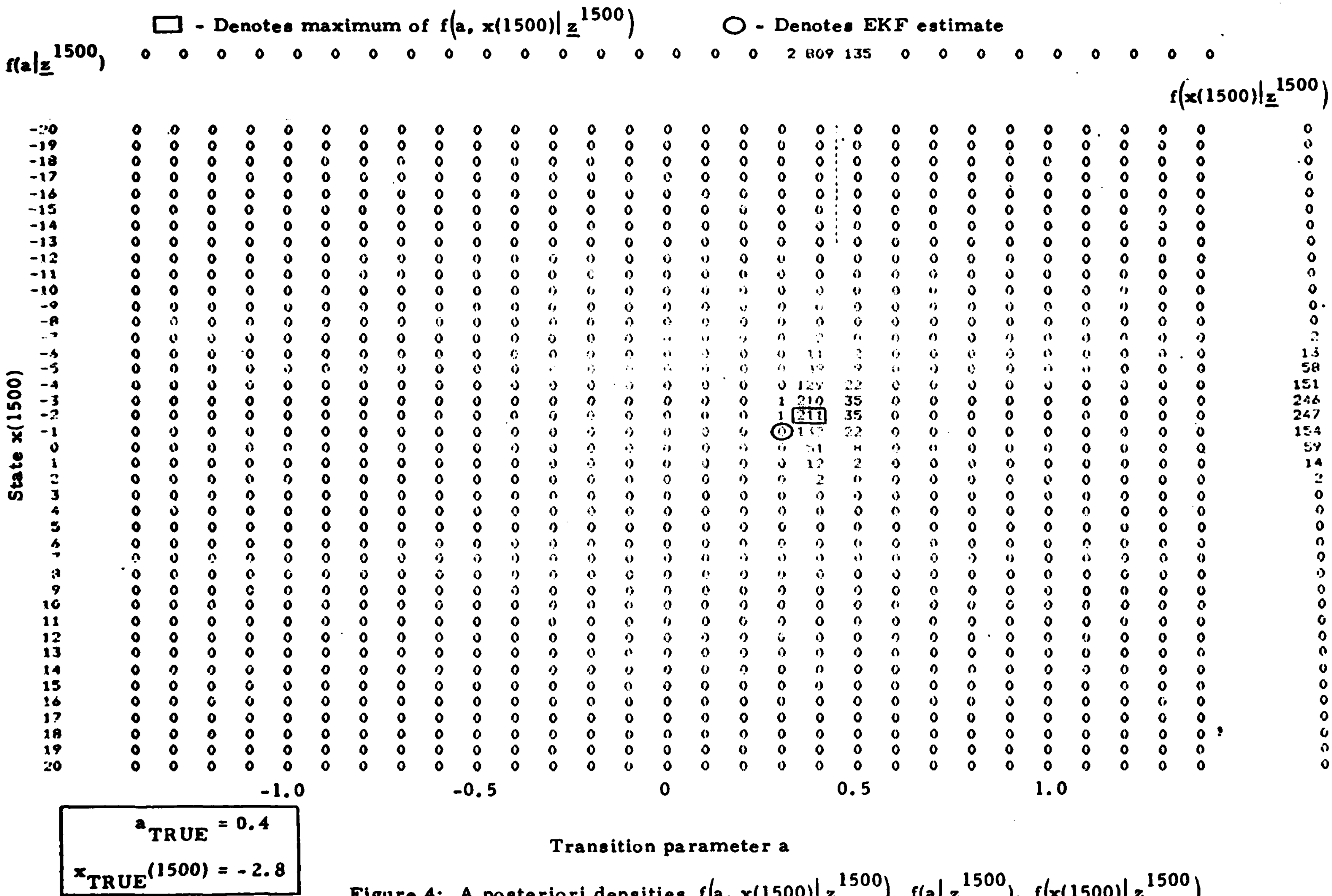
Transition parameter a

$a_{TRUE} = 0.4$

$x_{TRUE}(1000) = 1.6$

Figure 3: A posteriori densities $f\left(a, x(1000) \mid \underline{z}^{1000}\right)$, $f\left(a \mid \underline{z}^{1000}\right)$, $f\left(x(1000) \mid \underline{z}^{1000}\right)$

Note: Density values have been scaled by a factor of 1000.




Transition parameter a

Figure 4: A posteriori densities $f\left(a, x(1500) \mid \underline{z}^{1500}\right)$, $f\left(a \mid \underline{z}^{1500}\right)$, $f\left(x(1500) \mid \underline{z}^{1500}\right)$

Note: Density values have been scaled by a factor of 1000.

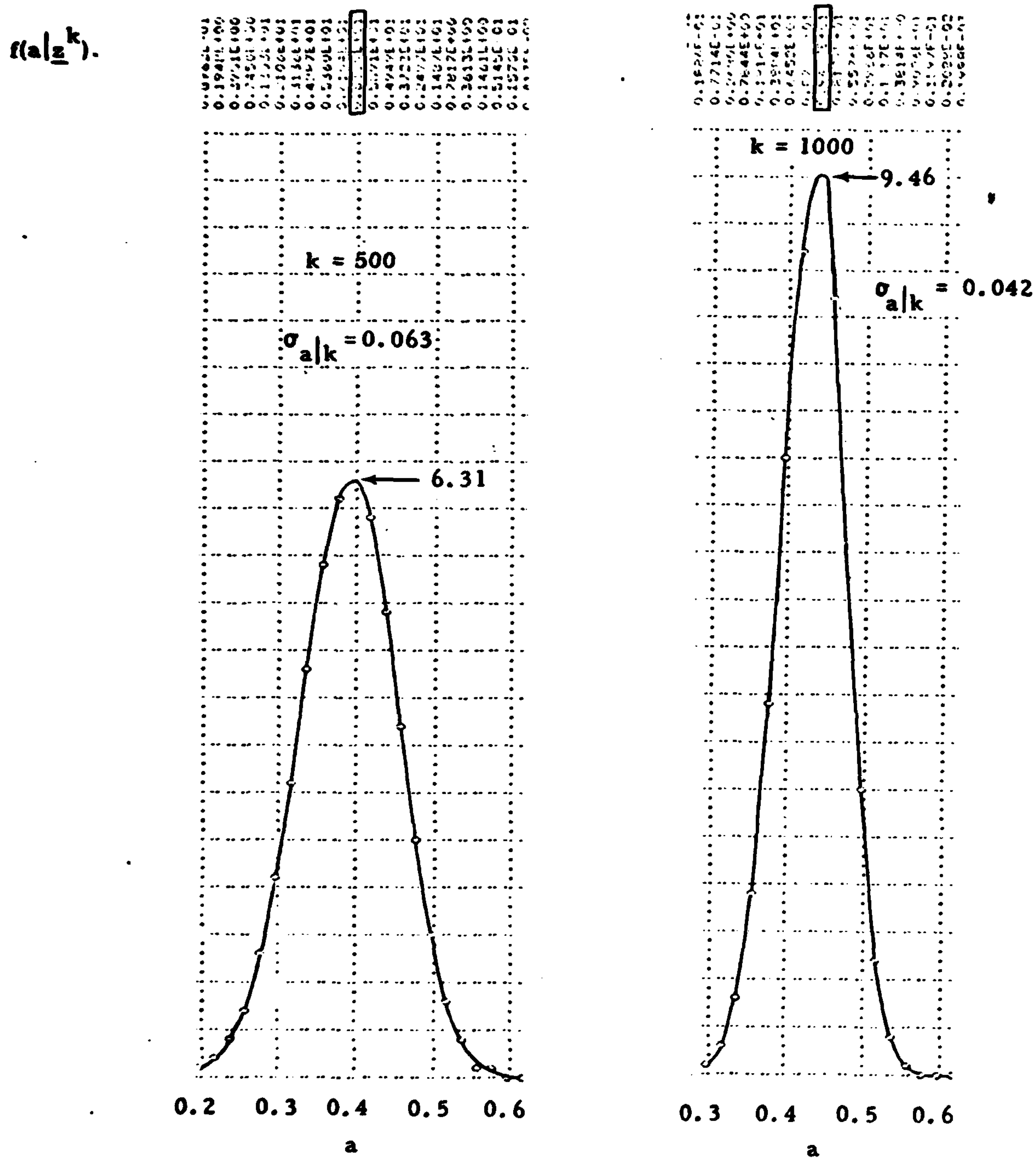



Figure 5: A posteriori density $f(a|\underline{z}^k)$; $a_{TRUE} = 0.4$

□ - Denotes maximum of $f(a, x(k) | \underline{z}^k)$ ○ - Denotes EKF estimate

$\boxed{a_{TRUE} = 0.9}$

**k = 500**

| State $x(k)$ \ a | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | $f(x(500) \mid \underline{z}^{500})$ |
|---|---|---|---|---|---|---|---|---|
| $f(a \mid \underline{z}^k)$ | 0 | 0 | 0 | 9 | 969 | 0 | 0 | |
| -20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -3 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 2 |
| -2 | 0 | 0 | 0 | 0 | 13 | 0 | 0 | 13 |
| -1 | 0 | 0 | 0 | 1 | 50 | 0 | 0 | 51 |
| 0 | 0 | 0 | 0 | 1 | 127 | 0 | 0 | 130 |
| 1 | 0 | 0 | 0 | 2 | 220 | 0 | 0 | 222 |
| 2 | 0 | 0 | 0 | 2 | $\boxed{247}$ | 0 | 0 | 249 |
| 3 | 0 | 0 | 0 | 2 | 183 | 0 | 0 | 185 |
| 4 | 0 | 0 | 0 | 1 | 89 | 0 | 0 | 90 |
| 5 | 0 | 0 | 0 | 0 | 29 | 0 | 0 | 29 |
| 6 | 0 | 0 | 0 | 0 | 6 | 0 | 0 | 6 |
| 7 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$\boxed{x_{TRUE}(500) = 2.32}$

**k = 1000**

| State $x(k)$ \ a | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | $f(x(1000) \mid \underline{z}^{1000})$ |
|---|---|---|---|---|---|---|---|---|
| $f(a \mid \underline{z}^k)$ | 0 | 0 | 0 | 0 | 1268 | 0 | 0 | |
| -20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -6 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| -5 | 0 | 0 | 0 | 0 | 10 | 0 | 0 | 10 |
| -4 | 0 | 0 | 0 | 0 | 44 | 0 | 0 | 44 |
| -3 | 0 | 0 | 0 | 0 | 131 | 0 | 0 | 131 |
| -2 | 0 | 0 | 0 | 0 | 255 | 0 | 0 | 255 |
| -1 | 0 | 0 | 0 | 0 | $\boxed{327}$ | 0 | 0 | 327 |
| 0 | 0 | 0 | 0 | 0 | 276 | 0 | 0 | 276 |
| 1 | 0 | 0 | 0 | 0 | 153 | 0 | 0 | 153 |
| 2 | 0 | 0 | 0 | 0 | 56 | 0 | 0 | 56 |
| 3 | 0 | 0 | 0 | 0 | 13 | 0 | 0 | 13 |
| 4 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 2 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$\boxed{x_{TRUE}(1000) = 0.65}$

**k = 1500**

| State $x(k)$ \ a | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | $f(x(1500) \mid \underline{z}^{1500})$ |
|---|---|---|---|---|---|---|---|---|
| $f(a \mid \underline{z}^k)$ | 0 | 0 | 0 | 0 | 3159 | 0 | 0 | |
| -20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -8 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 3 |
| -7 | 0 | 0 | 0 | 0 | 19 | 0 | 0 | 19 |
| -6 | 0 | 0 | 0 | 0 | 90 | 0 | 0 | 90 |
| -5 | 0 | 0 | 0 | 0 | 285 | 0 | 0 | 285 |
| -4 | 0 | 0 | 0 | 0 | 590 | 0 | 0 | 590 |
| -3 | 0 | 0 | 0 | 0 | $\boxed{805}$ | 0 | 0 | 805 |
| -2 | 0 | 0 | 0 | 0 | 723 | 0 | 0 | 723 |
| -1 | 0 | 0 | 0 | 0 | 427 | 0 | 0 | 427 |
| 0 | 0 | 0 | 0 | 0 | 166 | 0 | 0 | 166 |
| 1 | 0 | 0 | 0 | 0 | 43 | 0 | 0 | 43 |
| 2 | 0 | 0 | 0 | 0 | 7 | 0 | 0 | 7 |
| 3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 17 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 19 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$\boxed{x_{TRUE}(1500) = -2.13}$

Figure 6: A posteriori density $f(a, x(k) | \underline{z}^k)$, $f(a | \underline{z}^k)$, $f(x(k) | \underline{z}^k)$

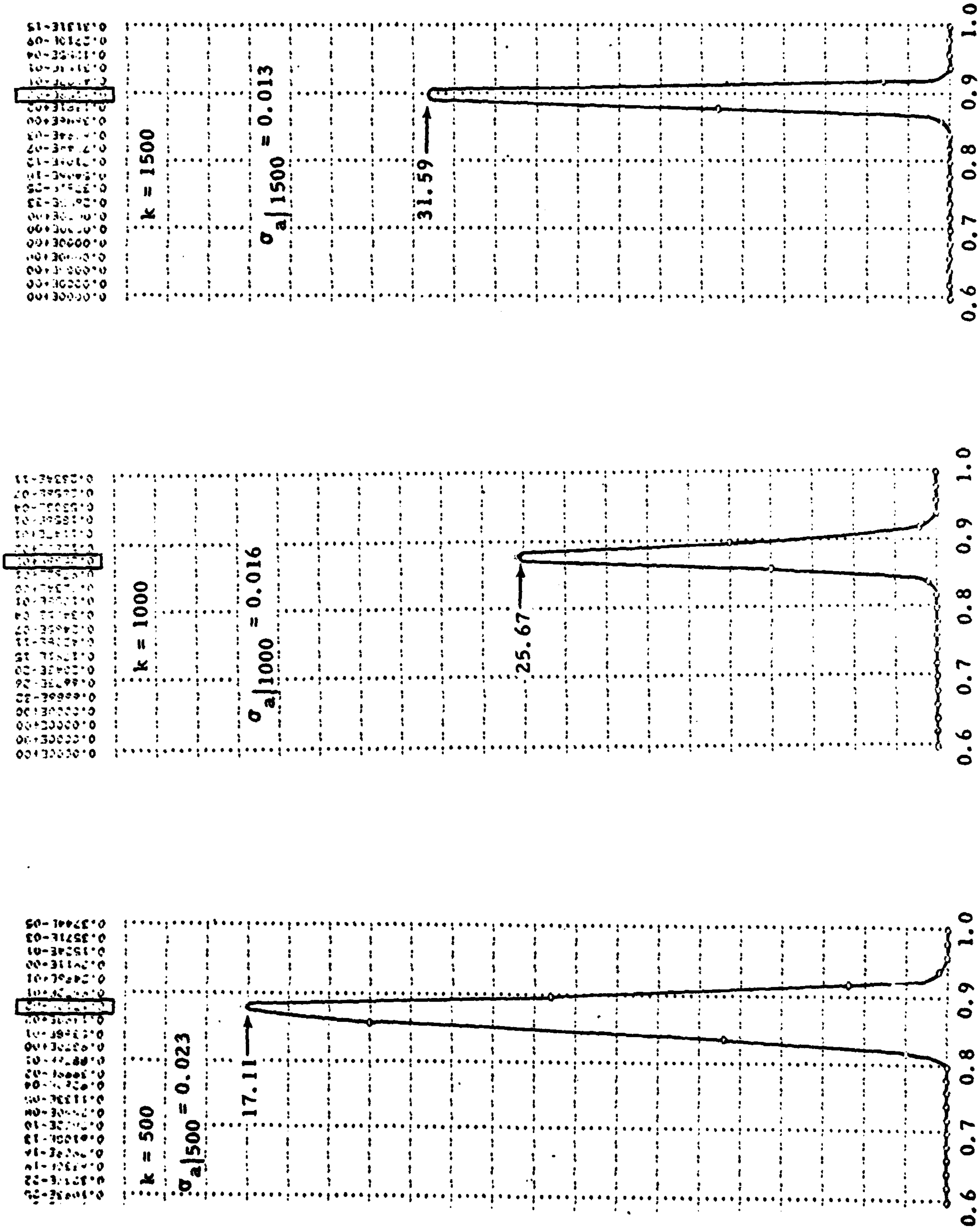



Figure 7: A posteriori density $f(a|\underline{z}^k)$; $a_{TRUE} = 0.9$

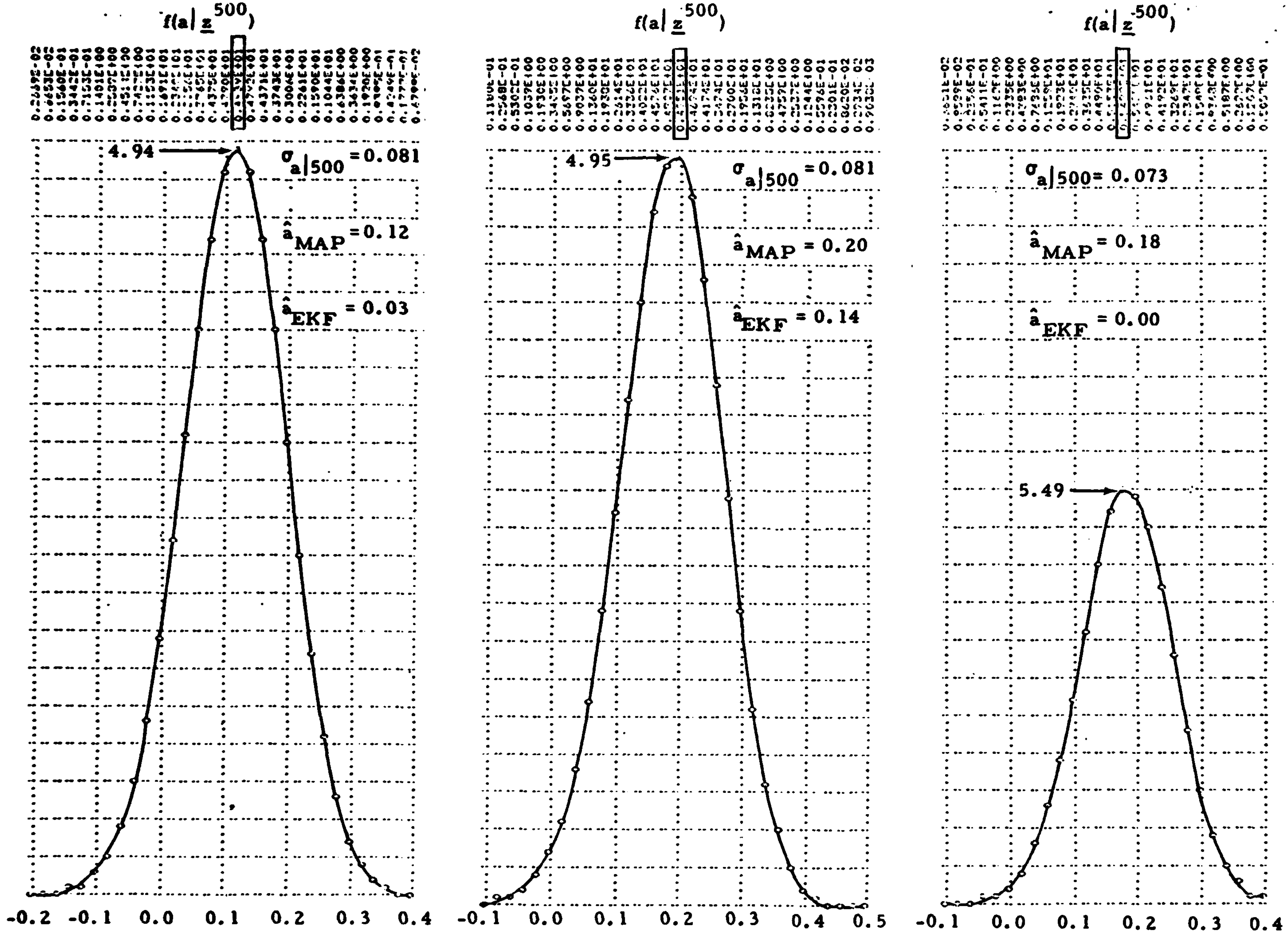



Figure 8: $f(a|\underline{z}^{500})$ for different noise realizations; $a_{TRUE} = 0.1$

$\boxed{\phantom{x}}$ - Denotes maximum of $f(a, x(500) \mid \underline{z}^{500})$

○ - Denotes EKF estimate

| State x(500) | | | | | | | | | $f(x \mid \underline{z}^{500})$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(a \mid \underline{z}^{500})$ | 0 | 0 | 15 | 340 | 603 | 44 | 0 | 0 | |
| -9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -7 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 2 |
| -6 | 0 | 0 | 0 | 4 | 8 | 1 | 0 | 0 | 13 |
| -5 | 0 | 0 | 1 | 17 | 34 | 3 | 0 | 0 | 55 |
| -4 | 0 | 0 | 2 | 49 | 91 | 7 | 0 | 0 | 149 |
| -3 | 0 | 0 | 4 | 85 | 152 | 11 | 0 | 0 | 252 |
| -2 | 0 | 0 | 4 | 91 | $\boxed{159}$ | 11 | 0 | 0 | 265 |
| -1 | 0 | 0 | 3 | 61 | 103 | 7 | 0 | 0 | 174 |
| 0 | 0 | 0 | 1 | 25 | 42 | 3 | 0X | 0 | 71 |
| 1 | 0 | 0 | 0 | 6 | 11 | 1 | 0 | 0 | 18 |
| 2 | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 3 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0.1 | | 0.3 | | 0.5 | | 0.7 | | |

$x_{TRUE}(500) = -1.44$ $\quad$ $u(t) \equiv 0$

| State x(500) | | | | | | | | $f(x \mid \underline{z}^{500})$ |
|---|---|---|---|---|---|---|---|---|
| $f(a \mid \underline{z}^{500})$ | 0 | 0 | 0 | 2002 | 0 | 0 | 0 | |
| -13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -11 | 0 | 0 | 0 | 5 | 0 | 0 | 0 | 5 |
| -10 | 0 | 0 | 0 | 31 | 0 | 0 | 0 | 31 |
| -9 | 0 | 0 | 0 | 130 | 0 | 0 | 0 | 130 |
| -8 | 0 | 0 | 0 | 333 | 0 | 0 | 0 | 333 |
| -7 | 0 | 0 | 0 | $\boxed{527}$ | 0 | 0 | 0 | 527 |
| -6 | 0 | 0 | 0 | 516 | 0 | 0 | 0 | 516 |
| -5 | 0 | 0 | 0 | 312 | 0 | 0 | 0 | 312 |
| -4 | 0 | 0 | 0 | 117 | 0 | 0 | 0 | 117 |
| -3 | 0 | 0 | 0 | 27 | 0 | 0 | 0 | 27 |
| -2 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 4 |
| -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | | 0.2 | | 0.4 | | 0.6 | | |

$x_{TRUE}(500) = -5.05$ $\quad$ Sinusoidal Input

| State x(500) | | | | | | | | $f(x \mid \underline{z}^{500})$ |
|---|---|---|---|---|---|---|---|---|
| $f(a \mid \underline{z}^{500})$ | 0 | 0 | 7 | 1330 | 0 | 0 | 0 | |
| -9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -7 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 2 |
| -6 | 0 | 0 | 0 | 15 | 0 | 0 | 0 | 15 |
| -5 | 0 | 0 | 0 | 67 | 0 | 0 | 0 | 67 |
| -4 | 0 | 0 | 1 | 190 | 0 | 0 | 0 | 191 |
| -3 | 0 | 0 | 2 | 330 | 0 | 0 | 0 | 332 |
| -2 | 0 | 0 | 2 | $\boxed{356}$ | 0 | 0 | 0 | 358 |
| -1 | 0 | 0 | 1 | 237 | 0 | 0 | 0 | 238 |
| 0 | 0 | 0 | 1 | 97 | 0 | 0 | 0 | 98 |
| 1 | 0 | 0 | 0 | 25 | 0 | 0 | 0 | 25 |
| 2 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 4 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | | 0.2 | | 0.4 | | 0.6 | | |

$x_{TRUE}(500) = -1.44$ $\quad$ Pulsed Input

| State x(500) | | | | | | | | | $f(x \mid \underline{z}^{500})$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(a \mid \underline{z}^{500})$ | 0 | 0 | 0 | 22 | 1046 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 3 |
| 3 | 0 | 0 | 0 | 1 | 21 | 0 | 0 | 0 | 22 |
| 4 | 0 | 0 | 0 | 2 | 81 | 0 | 0 | 0 | 83 |
| 5 | 0 | 0 | 0 | 5 | 191 | 0 | 0 | 0 | 196 |
| 6 | 0 | 0 | 0 | 6 | $\boxed{278}$ | 0 | 0 | 0 | 284 |
| 7 | 0 | 0 | 0 | 5 | 251 | 0 | 0 | 0 | 256 |
| 8 | 0 | 0 | 0 | 2 | 140 | 0 | 0 | 0 | 142 |
| 9 | 0 | 0 | 0 | 1 | 48 | 0 | 0 | 0 | 49 |
| 10 | 0 | 0 | 0 | 0 | 10 | 0 | 0 | 0 | 10 |
| 11 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0.0 | | 0.2 | | 0.4 | | 0.6 | | |

$x_{TRUE}(500) = 4.08$ $\quad$ Pseudo Random

Figure 9: A posteriori densities for different inputs

Note: Density for Gaussian white input is similar to pseudo random and is omitted.

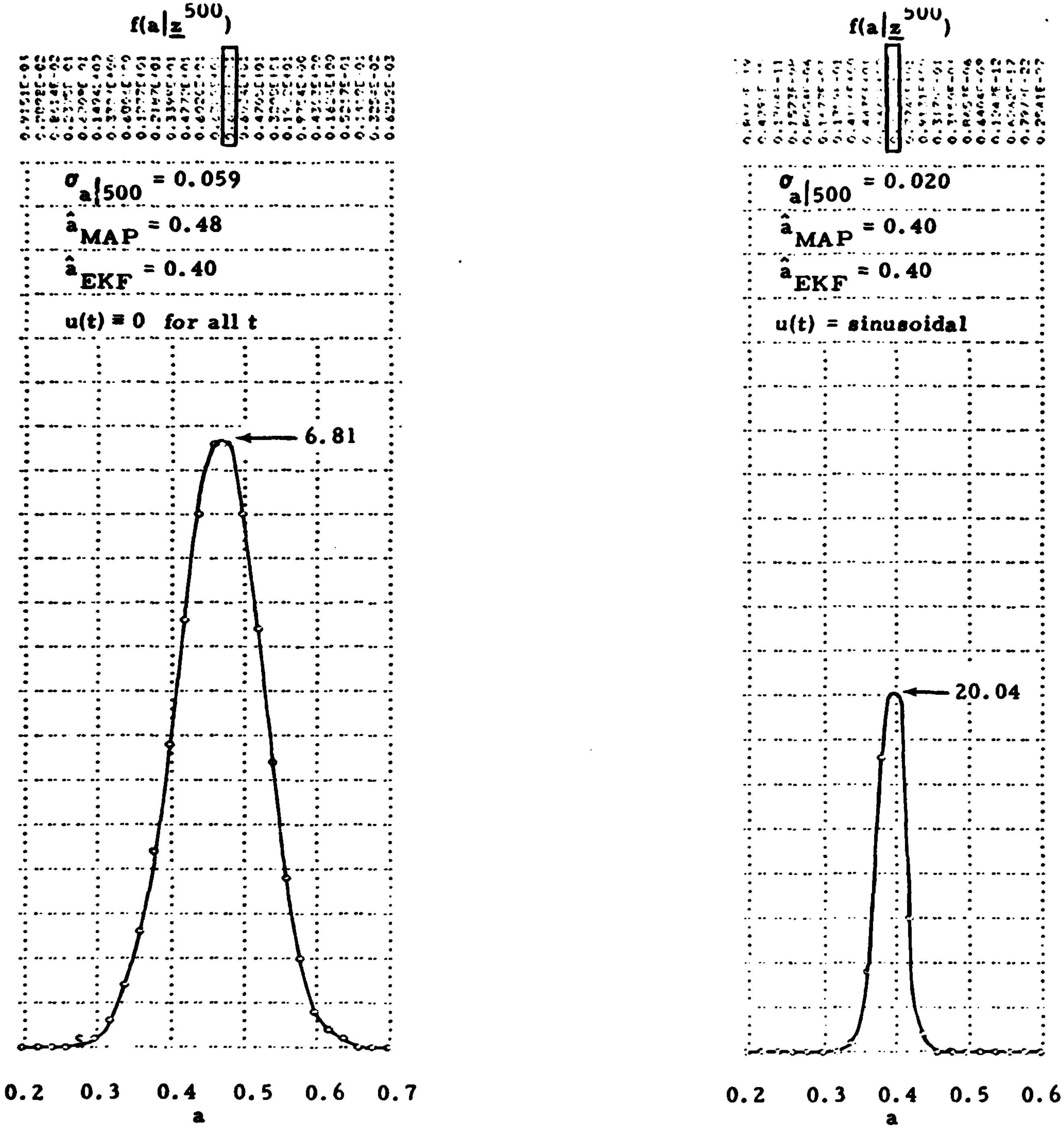



Figure 10: $f(a|z^{500})$ for zero input and for sinusoidal input

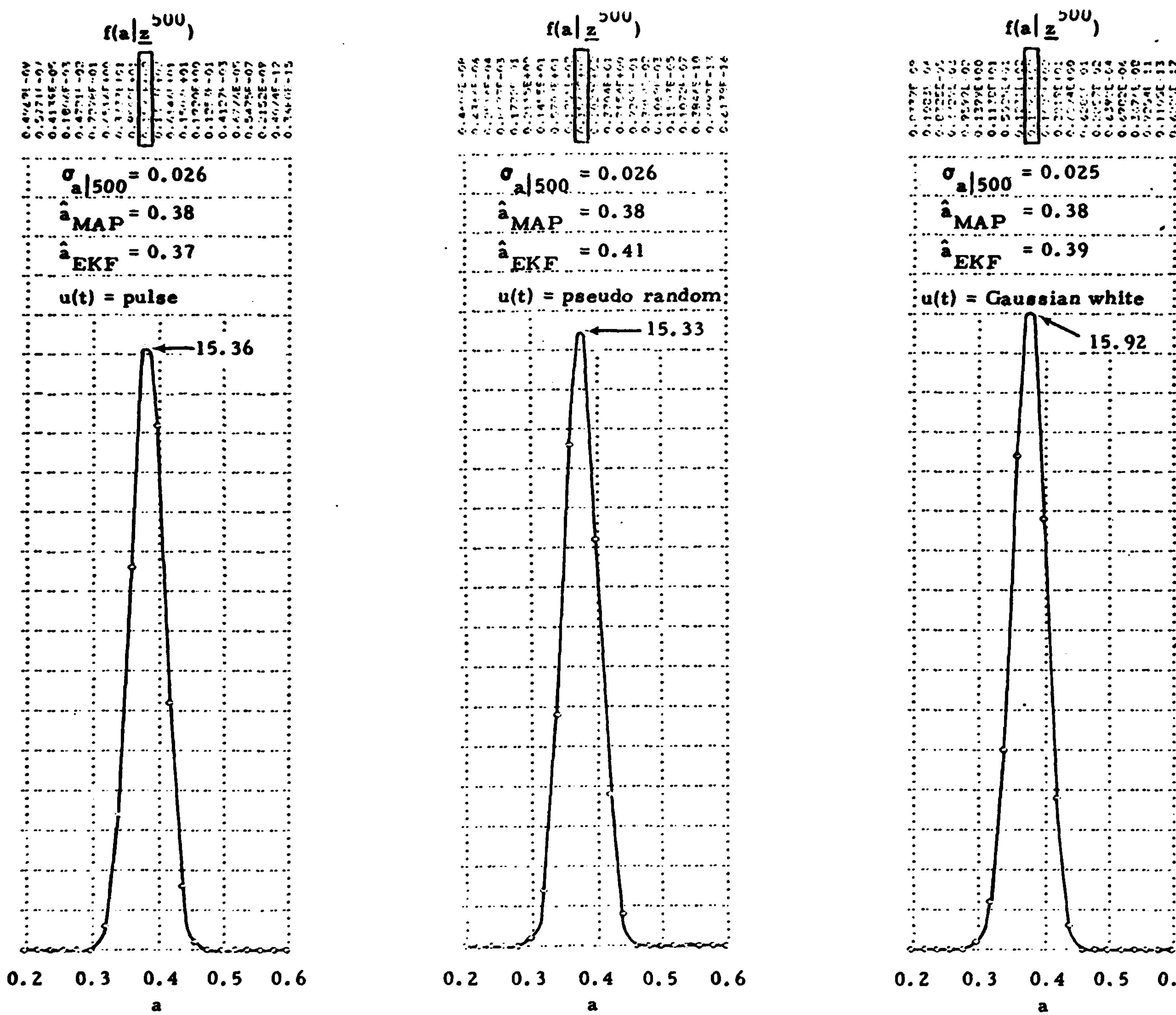



Figure 11: $f(a|\underline{z}^{500})$ for pulse, pseudo-random, Gaussian white inputs

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| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s) H.W. Sorenson | | 8. CONTRACT OR GRANT NUMBER(s) AFOSR 75-2839 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Applied Mechanics and Engr. Sciences University of California, San Diego La Jolla, California 92093 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A6 |
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Input Signal Synthesis
System Identification
Optimal Stochastic Control

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The research that has been supported by Air Force Research Grant AFOSR 75-2839 is described in the Final Scientific Report. This grant was initiated on June 1, 1975 and ended ~~on~~ May 31, 1980. This report provides a comprehensive review of accomplishments and a chronological bibliography of all publications resulting from ~~the~~ support provided by the grant.